

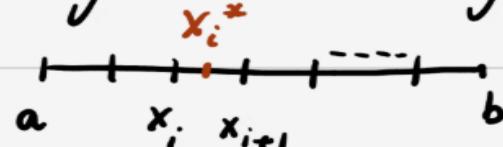
## Section 5.4 Work

If an object moves along a straight line, with position function  $S(t)$ , then the force on this object is given by  $F = m \frac{d^2 S}{dt^2}$  where  $m$  is the mass of the object in kg,  $t$  is in seconds, and  $F$  is in Newtons (N).

- If acceleration is constant (constant force), the work done is defined to be  $W = F \times d$ , where  $d$  is the distance traveled by the object.
  - $W$  is in Joules (J) when  $F$  (N) and  $d$  (m).
  - $W$  is in ft-lb when  $F$  (lb) and  $d$  (ft).

**Example 1** (a) Lifting a 2kg book off the floor 1.5m high requires  $W = F \times d = m \cdot g \times d = 2 \times 9.8 \times 1.5 = 29.4$  J  
(b) Lifting a 10 lb weight 8ft off the ground requires  $W = F \times d = 10 \times 8 = 80$  ft-lb

Now, what happens if the force is not constant? Say  $f(x)$  is the force acting on a body moving along the  $x$ -axis. Say the body moves from  $x=a$  to  $x=b$



Divide  $[a, b]$  into  $n$  sub-intervals  $[x_i, x_{i+1}]$ , and pick a point  $x_i^*$  from each sub-interval. Then we approximate the

force on  $[x_i, x_{i+1}]$  by  $f(x_i^*)$ . The Work done by moving the body from  $x_i$  to  $x_{i+1}$  is  $W_i = f(x_i^*) \cdot (x_{i+1} - x_i) = f(x_i^*) \cdot \Delta x$   
So, the total work  $W$  is given by

$$W = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f(x_i^*) \cdot \Delta x = \int_a^b f(x) dx \quad (\text{Riemann Sum})$$

**Example 2.** When an object is located  $x$  feet from the origin, a force of  $(x^3 + 3x^2)$  lbs acts on it. How much work is done in moving it from  $x=1$  to  $x=2$ ?

$$W = \int_1^2 f(x) dx = \int_1^2 (x^3 + 3x^2) dx = \left. \frac{x^4}{4} + x^3 \right|_1^2 = 12 - \frac{5}{4} = 10.75 \text{ ft-lb}$$

**Hooke's law:** The force required to maintain a spring stretched  $x$  units beyond its natural length is  $f(x) = k \cdot x$ , where  $k$  is the spring constant.

**Example 3** A force of 50N is required to hold a Spring that has been stretched from its natural length of 20 cm, to 25 cm. How much Work is done in stretching the spring from 25 cm to 30 cm?

First we find  $K$ :  $f = kx \Rightarrow k = f/x = 50/0.05 = 1000 \text{ N/m}$ . Therefore the force is  $f(x) = 1000x$ . With this, the work is given by

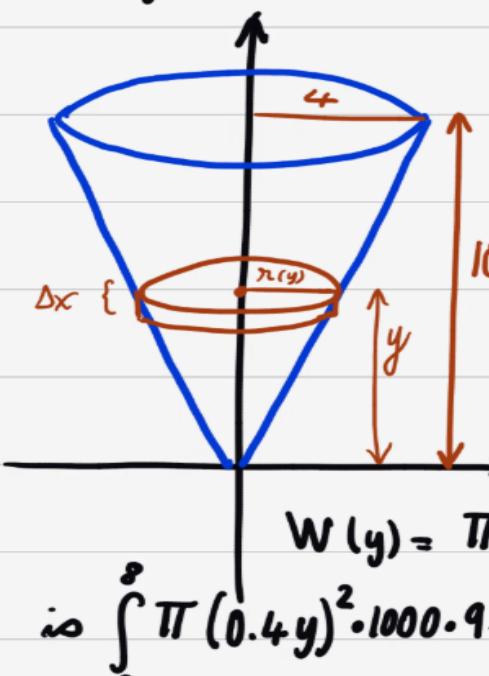
$$W = \int_{0.05}^{0.1} 1000x dx = 3.75 \text{ J.}$$

**Example 4** A 300 - ft long cable hangs vertically from the top of a tall building. How much work is required to lift the cable to the top? The cable weighs 3 lbs/ft.

0 Consider a small section of the cable of length  $\Delta x$ , located at  $x_i^*$  ft from the top; it weighs  $3 \cdot \Delta x$  lbs. The work done by lifting it to the top is  $W_i = 3 \cdot \Delta x \cdot x_i^*$  ft-lb. Therefore, the total work is

$$300 \quad W = \int_0^{300} 3x \, dx = 3x^2/2 \Big|_0^{300} = 135000 \text{ ft-lb.}$$

**Example 5** A tank has the shape of a downward pointing circular cone, of height 10m and radius 4m. It is filled with water to the height of 8m. How much work is done in pumping the water out of the tank, if the pump is placed at the top of the tank? (water density is 1000 Kg/m<sup>3</sup>)



Consider a small disk of water at height  $y$ . It has a circular shape, with radius  $r(y) = \frac{4}{10}y = 0.4y$  m (similar triangles). Its volume is  $V(y) = \pi(0.4y)^2 \cdot \Delta y$  m<sup>3</sup>

It weighs  $w(y) = \pi(0.4y)^2 \cdot \Delta y \cdot 1000 \cdot 9.8$  N

The work required to lift it  $(10-y)$  m is

$$W(y) = \pi(0.4y)^2 \cdot \Delta y \cdot 1000 \cdot 9.8 \cdot (10-y) \text{ J. Therefore, Total work}$$

$$\text{is } \int_0^8 \pi(0.4y)^2 \cdot 1000 \cdot 9.8 \cdot (10-y) \, dy = 3362827.797 \text{ J}$$